**Markov Chain Monte Carlo (MCMC)**

Markov Chain Monte Carlo (MCMC) methods are a class of algorithms used to sample from probability distributions when direct sampling is difficult. They are particularly useful for estimating the distribution of complex, high-dimensional spaces.

**Metropolis-Hastings Algorithm**

The Metropolis-Hastings algorithm is a specific type of MCMC method. It generates a sequence of sample values from a target probability distribution \( f(x) \), even if the distribution is known only up to a normalizing constant.

**Key Concepts:**

1. **Target Distribution**: The probability distribution we want to sample from.

2. **Proposal Distribution**: A distribution used to generate new sample candidates. In your problem, the proposal distribution is a normal distribution with mean equal to the current sample value and standard deviation .

3. **Acceptance Ratio**: The ratio that determines whether to accept or reject the proposed sample. For the Metropolis-Hastings algorithm:

If , the proposed sample is always accepted. If , the proposed sample is accepted with probability .

**Steps in the Algorithm**

1. **Initialization**: Start with an initial value .

2. **Iteration**

* Propose a new sample from the proposal distribution.
* Compute the acceptance ratio .
* Generate a uniform random number from .
* If , accept the new sample and set . Otherwise, reject the new sample and set .

The given target distribution is:

This is the probability density function of the Laplace distribution (also known as the double-exponential distribution), centred at 0 with a scale parameter of 1.

**Why Use the Logarithm?**

To avoid numerical errors, particularly underflow or overflow issues when dealing with very small or very large numbers, it is common to work with the logarithm of the acceptance ratio:

\[ \log r(x^\*, x\_{i-1}) = \log f(x^\*) - \log f(x\_{i-1}) \]

This ensures stability in computations.

**Outputs**

The algorithm returns a sequence of samples used to generate;

1. **Histogram and Kernel Density Plot**: Estimate the target distribution visually.

2. **Monte Carlo Estimates**: Compute the sample mean and standard deviation, which are estimates of the true mean and standard deviation of the target distribution.

**Summary of the Statistical Background**

- **MCMC Methods**: Useful for sampling from complex distributions.

- **Metropolis-Hastings Algorithm**: Generates a Markov chain that has the target distribution as its equilibrium distribution.

- **Laplace Distribution**: The given is a Laplace distribution with mean 0 and scale 1.

- **Logarithms for Stability**: Using logarithms in the acceptance ratio computation helps prevent numerical errors.

*In the coursework problem, the target distribution is known. This allows us to directly compare the generated samples to the true distribution. In practice, the Metropolis-Hastings algorithm is particularly useful when the target distribution is complex or unknown in full form. Often, we may know the distribution up to a normalizing constant or only have a proportional form of it. The algorithm allows us to generate samples that approximate this distribution, which is crucial for tasks like* ***Bayesian Inference*** *(a statistical method that updates the probability of a hypothesis as more evidence or information becomes available) where direct sampling is infeasible. By using a simpler proposal distribution and computing acceptance probabilities, we can iteratively build a sample that mirrors the target distribution. This method is essential for estimating parameters and understanding distributions in complex models, providing a way to visualize and compute statistics such as mean and standard deviation, even when the exact distribution is difficult to handle analytically.*